Thm. (Basic Laws on Limits). Assuming $\lim_{(x,y)\to(a,b)} f(x,y) = l_1$ and $\lim_{(x,y)\to(a,b)} g(x,y) = l_2$,

then we have,

(i) $\lim_{(x,y)\to(a,b)} f(x,y) \pm g(x,y) = \lim_{(x,y)\to(a,b)} f(x,y) \pm \lim_{(x,y)\to(a,b)} g(x,y) = l_1 \pm l_2$.

(ii) $\lim_{(x,y)\to(a,b)} cf(x,y) = c \lim_{(x,y)\to(a,b)} f(x,y) = cl_1$.

(iii) $\lim_{(x,y)\to(a,b)} f(x,y)g(x,y) = \left(\lim_{(x,y)\to(a,b)} f(x,y)\right)\left(\lim_{(x,y)\to(a,b)} g(x,y)\right) = l_1 l_2$.

(iv) $\lim_{(x,y)\to(a,b)} f(x,y)/g(x,y) = \dfrac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)} = \dfrac{l_1}{l_2}$ provided $\lim_{(x,y)\to(a,b)} g(x,y) = l_2 \neq 0$.

Thm. (Elementary Continuous functions)

All the polynomials, sine, cosine and exponential functions are continuous in their own arguments. Their composite functions are continuous as well.

Ex. $\lim_{(x,y)\to(2,2)} \dfrac{\cos(x^2+y^2)}{1-x^2-y^2} = \dfrac{\cos(8)}{1-4-4} = \dfrac{-\cos 8}{7}$ //.

Ex $\lim_{(x,y,z)\to(1,1,0)} \dfrac{xy-z}{\cos(xyz)} = \dfrac{(1)(1)-0}{\cos 0} = 1$ //.

Partial Derivatives

Definition: Given $z=f(x,y)$, we define

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}(x,y) = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y}(x,y) = f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

respectively to be the partial derivative of f with respect to x and partial deriv. of f wrt. y.

Essence: When we differentiate f partially with respective to x, it would be like holding y as a constant and vice versa.

Ex. Given $f(x,y) = x^2 + 4xy + y\sin x$

$$\frac{\partial f}{\partial x} = 2x + 4y + y\cos x, \quad \frac{\partial f}{\partial y} = 4x + \sin x$$

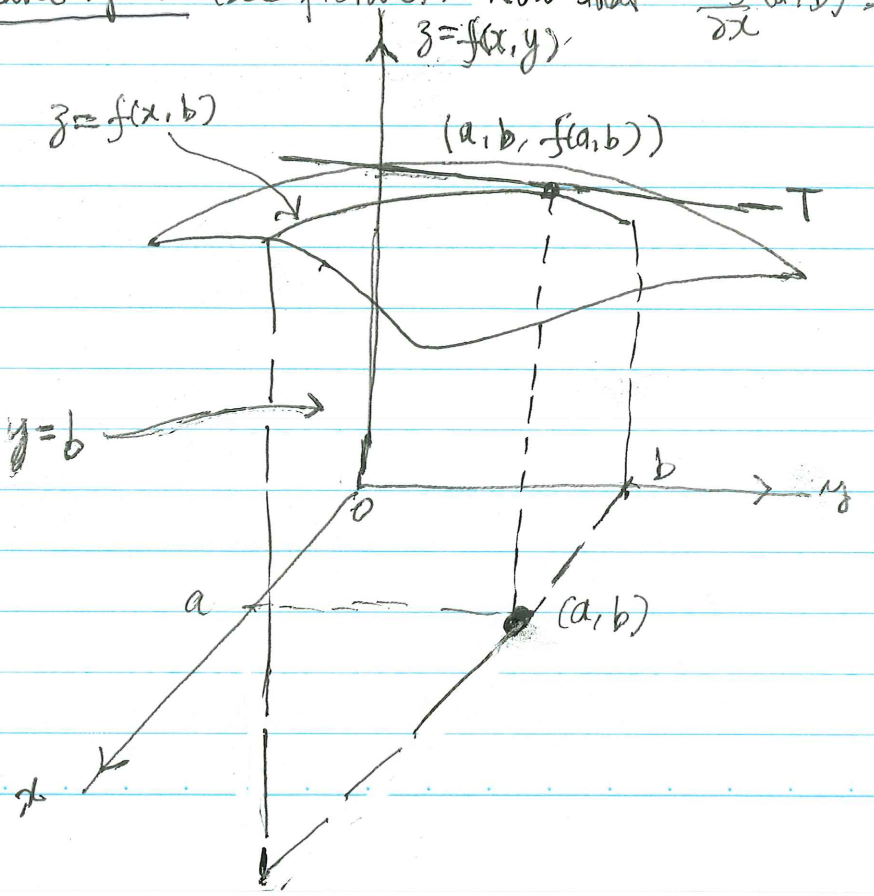
Remarks

(i) For function of three variables $f(x,y,z)$, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ are defined analogously.

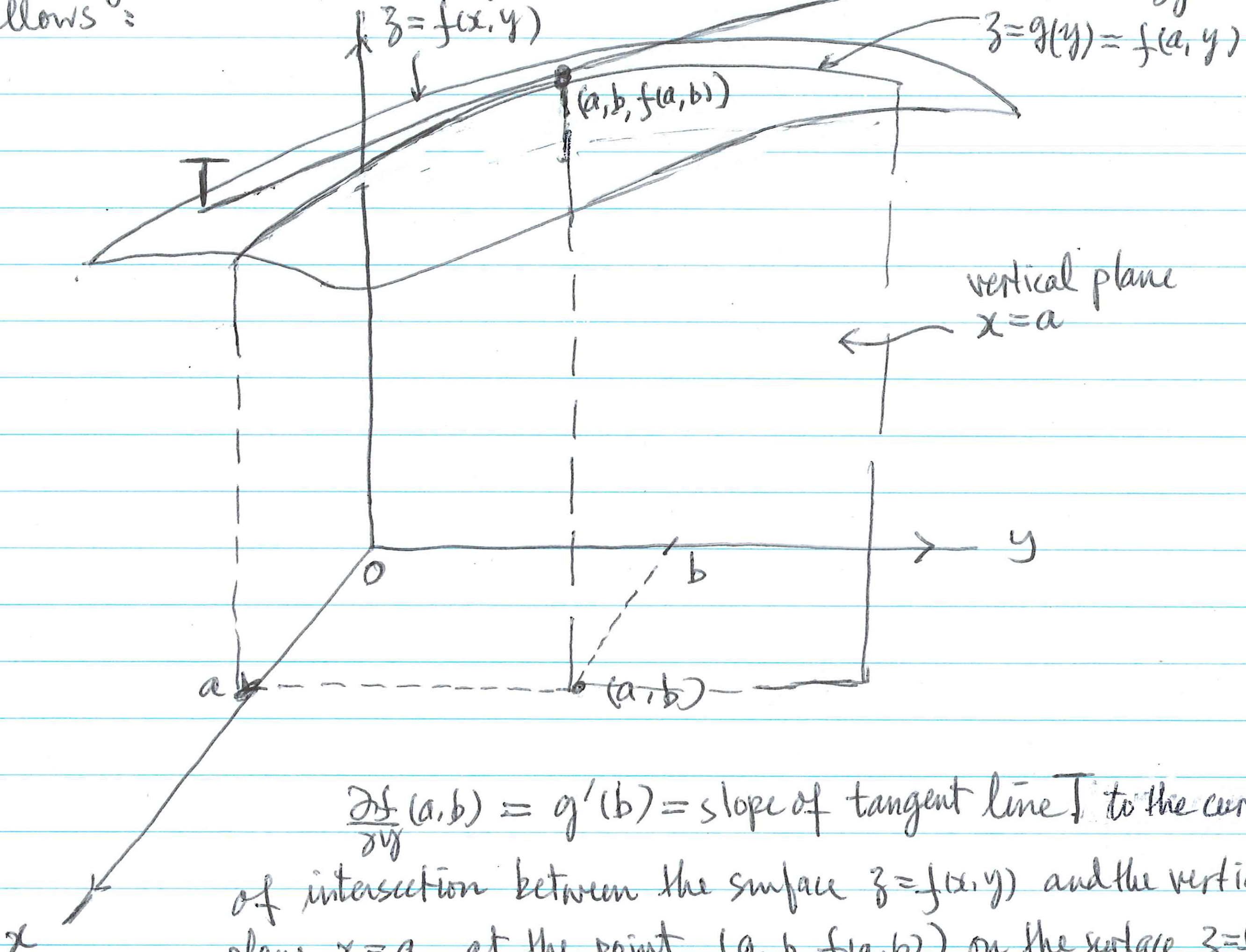
(ii) just like in the fct. of one variable case, $f'(x)$ is the rate of change of f with respect to x. $\frac{\partial f}{\partial x}$ is the rate of change of f with respect to x while the y variable is held fixed and so on.

The geometric meaning of $\frac{\partial f}{\partial x}(a,b)$ and $\frac{\partial f}{\partial y}(a,b)$ for $(a,b) \in$ domain of f

First we note that the graph of $z = f(x,y)$ is a surface in space, if we freeze the y variable to be at $y = b$, then $z = h(x) = f(x,b)$ becomes a function of x only. Its graph is the curve of intersection between the surface of $z = f(x,y)$ and the vertical plane $y = b$ (see picture). Now that $\frac{\partial f}{\partial x}(a,b) = h'(a)$ is the slope of the tangent line T to the curve of intersection $z = h(x) = f(x,b)$ at the point $(a, b, f(a,b))$ on the surface $z = f(x,y)$.



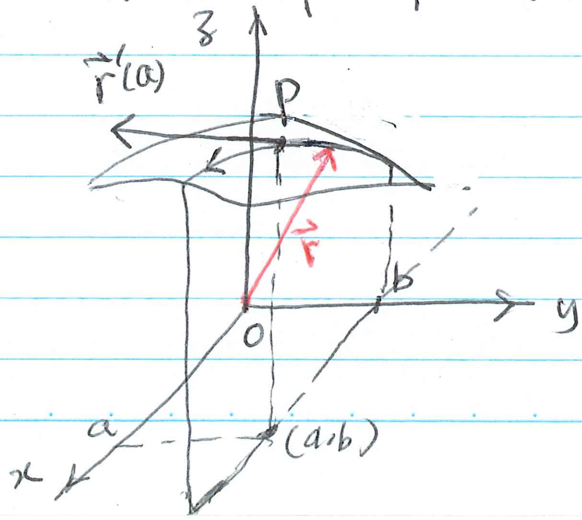
Similarly, we also have the geometric picture for the meaning of $\frac{\partial f}{\partial y}(a,b)$ as follows:



$\frac{\partial f}{\partial y}(a,b) = g'(b) =$ slope of tangent line T to the curve of intersection between the surface $z=f(x,y)$ and the vertical plane $x=a$ at the point $(a,b,f(a,b))$ on the surface $z=f(x,y)$.

Tangent Plane to the surface $z=f(x,y)$ at the point $(a,b,f(a,b))$ on the surface.

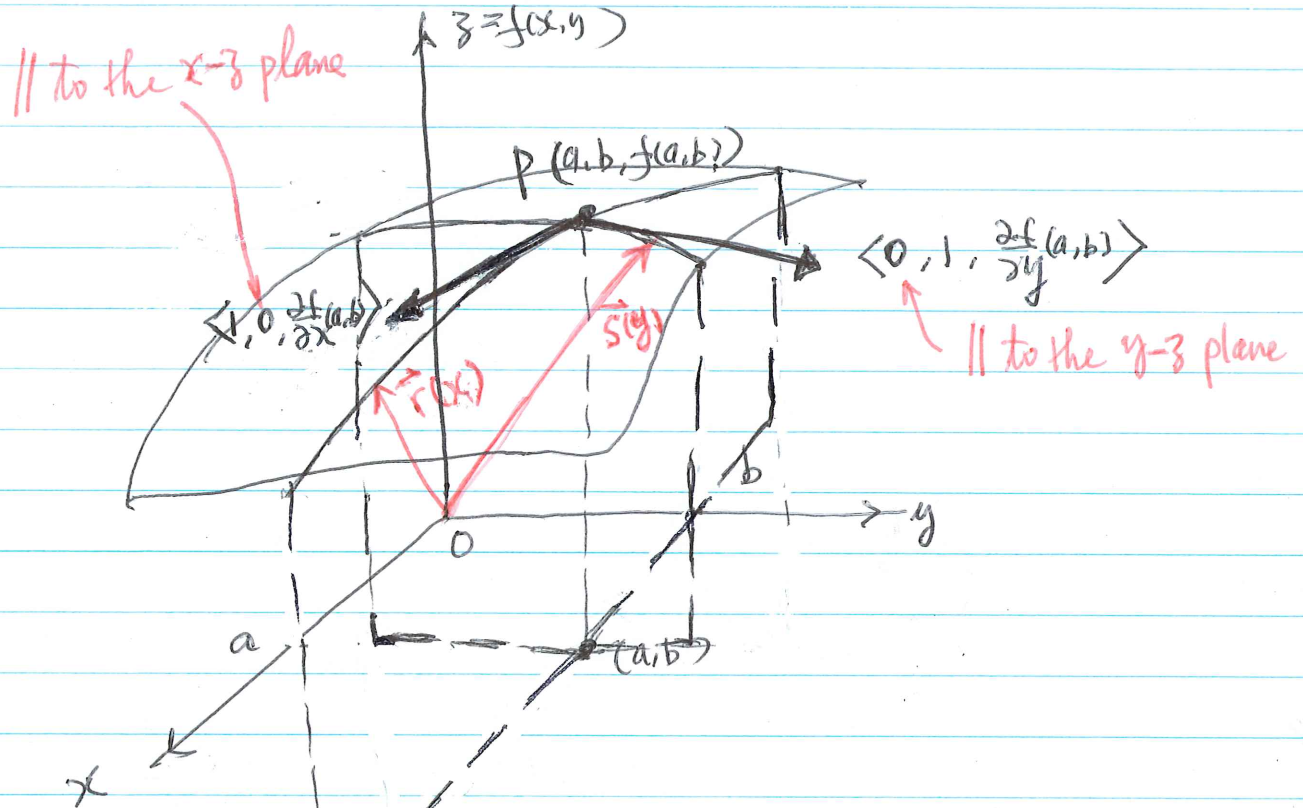
We once more define $\vec{r}(x)=\langle x, b, f(x,b)\rangle$ by freezing y at b, this is a vector-valued function which is a trajectory in space that coincides with the curve of intersection between $z=f(x,y)$ and the vertical plane $y=b$. Here x takes over the role of the parameter t. As a result,



$$\vec{r}'(a) = \langle 1, 0, \tfrac{\partial f}{\partial x}(a,b)\rangle$$

is tangential to the trajectory (being the velocity vector) and hence to the surface $z=f(x,y)$ at the point $P(a,b,f(a,b))$ on the surface.

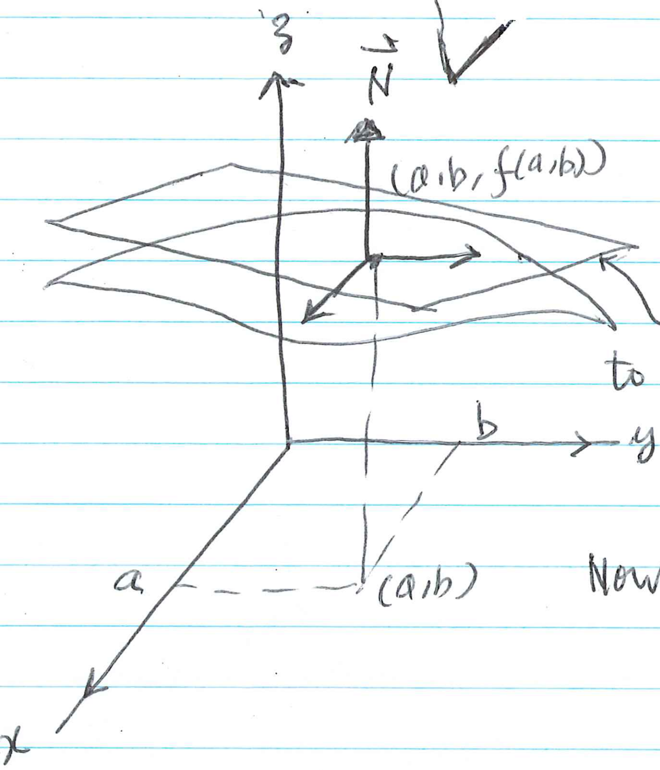
Similarly, by considering $\vec{s}(y) = \langle a, y, f(a,y) \rangle$, we could obtain another tangent ~~vector~~ vector $\langle 0, 1, \frac{\partial f}{\partial y}(a,b) \rangle$ to the surface at $(a, b, f(a,b))$



We could therefore take

$$\vec{N} = \langle 1, 0, \tfrac{\partial f}{\partial x}(a,b) \rangle \times \langle 0, 1, \tfrac{\partial f}{\partial y}(a,b) \rangle$$

to be a normal vector to the tangent plane (see picture).



tangent plane to the surface $z = f(x,y)$ at the pt. $(a, b, f(a,b))$ on the surface

Now that $\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial x}(a,b) \\ 0 & 1 & \frac{\partial f}{\partial x}(a,b) \end{vmatrix}$

$$= \langle -\tfrac{\partial f}{\partial x}(a,b), -\tfrac{\partial f}{\partial y}(a,b), 1 \rangle.$$

Therefore, equation of tangent plane at $(a,b,f(a,b))$ is given by

$$-\frac{\partial f}{\partial x}(a,b)(x-a) - \frac{\partial f}{\partial y}(a,b)(y-b) + (z - f(a,b)) = 0$$

alternatively, $\boxed{z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)}$ //

Ex Find equations of the tangent plane and the normal line to the surface $z = f(x,y) = e^{-x^2-y^2}$ at the point $P(1,1,e^{-2})$

Solution: $\frac{\partial f}{\partial x} = -2xe^{-x^2-y^2}$, $\frac{\partial f}{\partial y} = -2ye^{-x^2-y^2}$

$\vec{N} = \left\langle -\frac{\partial f}{\partial x}(1,1), -\frac{\partial f}{\partial y}(1,1), 1 \right\rangle = \left\langle -2e^{-2}, -2e^{-2}, 1 \right\rangle$ is a normal vector to the surface at the pt. P.

Hence, equations of tangent plane & normal line are respectively given by

$$-2e^{-2}(x-1) - 2e^{-2}(y-1) + (z - e^{-2}) = 0$$

alternatively, $z = e^{-2} + 2e^{-2}(x-1) + 2e^{-2}(y-1)$ //

and
$$\begin{cases} x = 1 - 2e^{-2}t \\ y = 1 - 2e^{-2}t \\ z = e^{-2} + t \end{cases} \quad -\infty < t < \infty.$$
//

Examples for partial derivatives of functions of more than 2 variables

Ex $g(x,y,u,v) = e^{ux}\sin(vy)$, then

$\frac{\partial g}{\partial x} = ue^{ux}\sin(vy)$, $\frac{\partial g}{\partial y} = ve^{ux}\cos vy$, $\frac{\partial g}{\partial u} = xe^{ux}\sin vy$, $\frac{\partial g}{\partial v} = ye^{ux}\cos vy$

Ex $f(x,y,z) = 4y^3\sin(x^2+4z^2y^3)$

$\frac{\partial f}{\partial x} = 4y^2\cos(x^2+4z^2y^2)(2x)$, $\frac{\partial f}{\partial y} = 8y\sin(x^2+4z^2y^3) + 4y^2\cos(x^2+4z^2y^3)(12z^2y^2)$

$\frac{\partial f}{\partial z} = 4y^2\cos(x^2+4z^2y^2)(8z)$ //

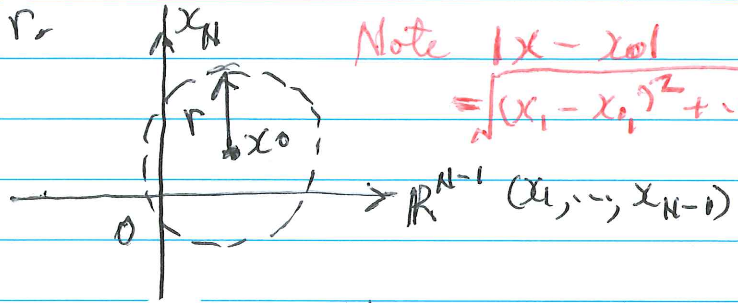
Higher Order Derivatives

f_{xy} or $\frac{\partial^2 f}{\partial y \partial x} = (f_x)_y$, f_{yx} or $\frac{\partial^2 f}{\partial x \partial y} = (f_y)_x$, $f_{xzy} = \frac{\partial^3}{\partial y \partial z \partial x} = ((f_x)_z)_y$

$\frac{\partial^3 f}{\partial y \partial x^2}$ or $f_{xxy} = (f_{xx})_y$. KEY: Always differentiate with respect to the variable nearest to f first.

Excursus: Basic Point Set Topology in $\mathbb{R}^N$

Definition: Let $x_0 = (x_{0_1}, x_{0_2}, \cdots, x_{0_N}) \in \mathbb{R}^N$, $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$ and $r > 0$, we define $B_r(x_0) = \{x \in \mathbb{R}^N \mid |x - x_0| < r\}$ to be the open ball centered at x_0 with radius r.



Note $|x - x_0|$
$= \sqrt{(x_1 - x_{0_1})^2 + \cdots + (x_N - x_{0_N})^2}$

We also have $\overline{B_r(x_0)} = \{x \in \mathbb{R}^N \mid |x - x_0| \le r\}$ to be a closed ball.

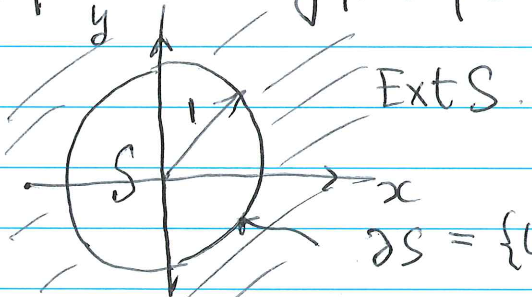
Defn. Let $S \subseteq \mathbb{R}^N$, $x_0 \in S$ is said to be an interior point of S if there exists $\varepsilon > 0$ sufficiently small such that $B_\varepsilon(x_0) \subset S$.
Consequently, we define $\text{Int}(S) = \{x \in \mathbb{R}^N \mid x \text{ is an interior point of } S\}$.

Defn. $x^* \in \mathbb{R}^N$ is an exterior point of S if there exists $\varepsilon > 0$ sufficiently small such that $B_\varepsilon(x^*) \subset \mathbb{R}^N \setminus S$ (i.e. $B_\varepsilon(x^*)$ lies outside S).
Analogously, we define $\text{Ext}(S) = \{x^* \in \mathbb{R}^N \mid x^* \text{ is an exterior point of } S\}$.

Definition $x_b \in \mathbb{R}^N$ is said to be a boundary point of S if for every $\varepsilon > 0$, we have $B_\varepsilon(x_b) \cap S \neq \phi$ as well as $B_\varepsilon(x_b) \cap \mathbb{R}^N \setminus S \neq \phi$.
Now we define $\partial S = \{x \in \mathbb{R}^N \mid x \text{ is a boundary point of } S\}$ to the boundary of S.

Ex $S = \{(x, y) \mid x^2 + y^2 < 1\}$

$\text{Int}\, S = S$.



$\partial S = \{(x, y) \mid x^2 + y^2 = 1\}$

Defn. $S \subseteq \mathbb{R}^N$, S is said to be an open set if $\forall x \in S$, $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subset S$. Further, S is closed if $\mathbb{R}^N \setminus S$ is open.

Remark:

1° S is open iff $S = \text{Int}(S)$

2° S is closed iff $S = \text{Int}\, S \cup \partial S$.

Ex. $f(x,y,t) = e^{-kt^2}\sin(x^2y^3)$, find $\frac{\partial^3}{\partial x \partial y \partial t} f$ and $\frac{\partial^3 f}{\partial y \partial t^2}$

$f_t = -2kte^{-kt^2}\sin(x^2y^3)$, $(f_{ty}) = -2kte^{-kt^2}\cos(x^2y^3)(3y^2) = -6kte^{-kt^2}y^2\cos(x^2y^3)$

$\therefore f_{tyx} = -6y^2kte^{-kt^2}(-\sin(x^2y^3))(2x)$

$= 12xy^2kte^{-kt^2}\sin(x^2y^3)$ //

$f_t = -2kte^{-kt^2}\sin(x^2y^3)$, $f_{tt} = \left(-2ke^{-kt^2} + 4k^2t^2e^{-kt^2}\right)\sin(x^2y^3)$

$(f_{tt})_y = \left(-2ke^{-kt^2} + 4k^2t^2e^{-kt^2}\right)(3y^2)\cos(x^2y^3)$ //

Th^m (Clairaut Theorem) Two partial derivatives involving the same variables are equal to each other at a point regardless of their order of differentiation if they are both continuous over an open set U in the domain of f containing the point.

Ex If f_{xxy} and f_{xyx} both exist and are continuous over an open set U in the domain of f, then $f_{xxy} = f_{xyx}$ over U.

Ex Given $f(r,s,t) = (1+r^2)e^{st}$ verify that $f_{rs} = f_{sr}$

Solution:

$f_r = 2re^{st}$, $f_s = t(1+r^2)e^{st}$

$\Rightarrow f_{rs} = 2rte^{st}$, $f_{sr} = 2rte^{st}$

Both continuous and equal //

Corollary: If f is continuously differentiable up to order n (i.e. all the partial derivatives of up to order n exist and are continuous, then the order of differentiation up to order n is irrelevant, any partial derivatives involving the same variables up to order n are the same regardless of the order of differentiation.

Ex. Determine if there is a twice continuously differentiable function $f(x,y)$ such that $f_x = \cos x \sin y$, $f_y = \sin x \cos y + 4$

Solution:

A preliminary test is to check if $f_{xy} = f_{yx}$ (for if such a f exists, we must have $f_{xy} = f_{yx}$).

Now that, $f_x = \cos x \sin y \Rightarrow f_{xy} = \cos x \cos y$

$f_y = \sin x \cos y + 4 \Rightarrow f_{yx} = \cos x \cos y$

$\therefore$ It is possible.

To recover f, we could integrate f_x partially with respect to x,

$$f(x,y) = \int f_x \, dx = \int \cos x \sin y \, dx = \sin y \int \cos x \, dx = \sin y \sin x + g(y)$$

(since we are integrating partially with respect to x, the arbitrary constant is a function of y)

Now that differentiate $f(x,y) = \sin y \sin x + g(y)$ with respect to y to get

$$f_y(x,y) = \cos y \sin x + g'(y)$$

But given $f_y = \sin x \cos y + 4$ $\therefore g'(y) = 4 \Rightarrow g(y) = \int 4 \, dy = 4y + C$

Thus, $f(x,y) = \sin x \sin y + 4y + C$ //.

Ex. Determine if there is a twice continuously differentiable $f(x,y,z)$ such that

$$\frac{\partial f}{\partial x} = z\cos y - y\cos xy, \quad \frac{\partial f}{\partial y} = -xz\sin y + 4y^3 - x\cos xy, \quad \frac{\partial f}{\partial z} = x\cos y + 4$$

(i) First of all, derive a preliminary test or necessary condition for the existence of such a f base on the Clairaut's Theorem, Carry out the test on the given f_x, f_y and f_z.

(ii) If the result of the test from part (i) is affirmative, recover $f(x,y,z)$ by integrating partially.

Solution:

(i) Base on Chairaut's Theorem, if such twice continuously differentiable f exists, we must have

$$f_{xy} = f_{yx}, \quad f_{xz} = f_{zx}, \quad f_{yz} = f_{zy}.$$

Indeed,

$$f_{xy} = -z\sin y - \cos xy + xy\sin xy = f_{yx}$$

$$f_{xz} = \cos y = f_{zx}$$

$$f_{yz} = -x\sin y = f_{zy}$$

Hence the preliminary test is verified.

(ii) Integrate $f_y = -xz\sin y + 4y^3 - x\cos xy$ partially with respect to y, we have

$$f(x,y,z) = \int -xz\sin y + 4y^3 - x\cos xy \, dy$$

$$= xz\cos y + y^4 - \sin xy + g(x,z)$$

Now that this $\Rightarrow$

$$f_z = x\cos y + \frac{\partial g(x,z)}{\partial z}$$, on comparing with the given $f_z = x\cos y + 4$

$$\Rightarrow \frac{\partial g(x,z)}{\partial z} = 4 \Rightarrow g(x,z) = \int 4\,dz + h(x) = 4z + h(x)$$

$$\Rightarrow f(x,y,z) = xz\cos y + y^4 - \sin xy + 4z + h(x)$$

Finally, differentiating this partially with respect to x,

$f_x = z\cos y - y\cos xy + h'(x)$ comparing with the given f_x, $h'(x) = 0$

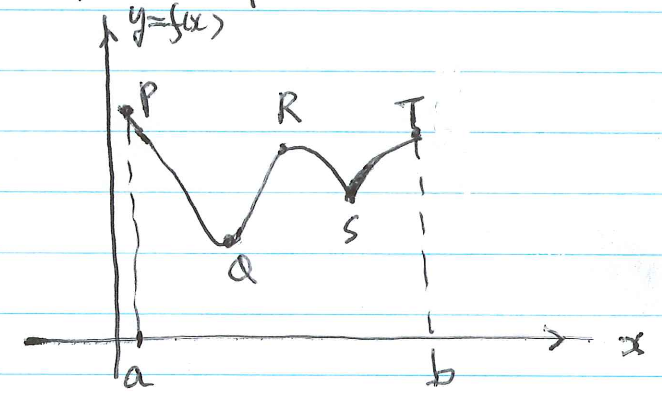
$\Rightarrow h(x) = C$ (a constant).

Hence,

$$f(x,y,z) = xz\cos y + y^4 - \sin xy + 4z + C \quad //$$

Optimization of functions of several variables.

Recall function of one variable case: $y = f(x)$, $a \le x \le b$.



P = Absolute/Global maximum (the highest pt.).

Q = Absolute/Global minimum (" lowest pt.).

S = local minimum

R, T = local maximum

Definition: For any point $c \in (a, b)$, it is said to be critical point of f iff either (i) $f'(c) = 0$ or (ii) $f'(c)$ does not exist.

Thm (Fundamental theorem for optimization) Let f be a continuous function over (a, b), we have

(i) If there is a local max or local min. at $c \in (a, b)$, then c is a critical point of f.

(ii) If f is continuous on $[a, b]$, then f attains an absolute maximum and an absolute minimum over $[a, b]$. Further, the abs. max. and abs. min. are being attained at either a critical point inside (a, b) or at the end pts.

Consider now the case of function of 2 variables:

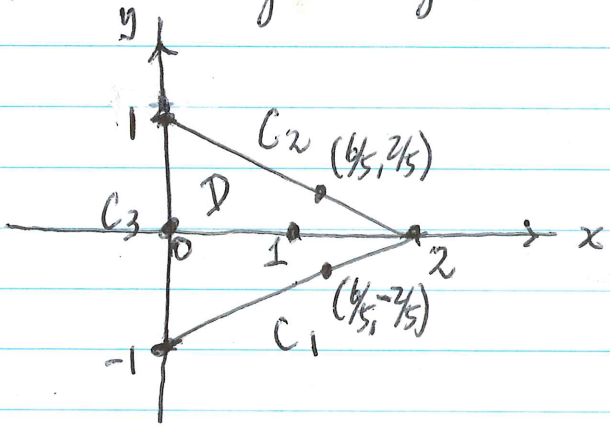
$$z = f(x, y), \quad (x, y) \in D$$

D is the domain of f, an open connected set in the xy plane with boundary ∂D. Exactly the same theory could be carried over.

Def'n. Given f continuous over D. $(a,b)\in D$ is said to be a critical point of f either (i) $\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0$ or (ii) one of the $\frac{\partial f}{\partial x}(a,b)$, $\frac{\partial f}{\partial y}(a,b)$ fails to exist.

Theorem: Given $z = f(x,y)$ a continuous function over D, if f attains a local minimum or a local maximum at $(a,b)\in D$, then (a,b) is a critical point of f. Further, if D is bounded and f is continuous over $\overline{D} = D\cup\partial D$ (i.e. f is continuous up to ∂D the boundary of D), then f attains an absolute max. and an absolute min. over $\overline{D}$, which is either a critical point or a boundary point.

Ex. Find the absolute max and the absolute min. of $f(x,y) = x^2+y^2-2x$, $(x,y)\in D$ where D is the triangular region as shown below.



Solution:

$f(x,y)$ is a polynomial in x & y, so it is continuous, further D is a bounded domain, therefore, the hypotheses of the theorem are satisfied.

Let us first figure out all the critical points in D by setting

$$\frac{\partial f}{\partial x} = 2x-2=0 \quad\text{and}\quad \frac{\partial f}{\partial y} = 2y = 0 \Rightarrow (1,0) \text{ is a critical pt.}$$

Since the abs. max. and abs. min. could also be attained at ∂D, we need to find out the max & min along C_1, C_2 and C_3.

Along C_1, $y = \frac{x}{2}-1$, $0\le x\le 2 \Rightarrow f(x,y) = x^2 + (\frac{x}{2}-1)^2 - 2x = \frac{5}{4}x^2 - 3x + 1$, $0\le x\le 2$.

Setting $f'(x) = \frac{5}{2}x - 3 = 0 \Rightarrow x = 6/5 \therefore (6/5, -2/5)$ is a critical pt. along C_1.

Along C_2, $y = -\frac{x}{2}+1$, $0\le x\le 2 \Rightarrow (6/5, 2/5)$ is a critical point along C_2.

Along C_3, $x=0$, $f(x,y) = y^2$, $-1\le y\le 1 \Rightarrow (0,0)$ is a critical pt. along C_3.

Finally, we must also take into consideration the end points of C_1, C_2 & C_3 which amounts to the corner points $(0, \pm 1)$, $(2,0)$. Thus, there are all-together 7 points where the abs. max & abs. min of f could be attained.
Evaluating f at those pts,

$$f(1,0) = -1, \; f(6/5, \pm 4/5) = -4/5, \; f(0,0) = 0, \; f(0,\pm 1) = 1 \text{ and } f(2,0) = 0.$$

Therefore $f(1,0) = -1$ is the abs. min. and $f(0, \pm 1) = 1$ are abs. max. //

Ex. (D is unbounded) Find the abs. maximum of $z = f(x,y) = \frac{8x^3}{3} + 4y^3 - x^4 - y^4$

Solution:

First we observe that as $x, y \to \pm\infty$, $f(x,y) \to -\infty$ ∴ there is no absolute minimum. Further, for (x,y) in the 1st quadrant which is near $(0,0)$, $f(x,y)$ is positive, hence there must be an absolute max. which is positive. At the abs. max, we have

$$\begin{cases} \frac{\partial f}{\partial x} = 8x^2 - 4x^3 = 0 \Rightarrow x = 0, 2 \\ \frac{\partial f}{\partial y} = 12y^2 - 4y^3 = 0 \Rightarrow y = 0, 3 \end{cases}$$

(note that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ always exist and there is no critical points of the 2nd type).

Thus, $(0,0)$, $(0,3)$, $(2,0)$ and $(2,3)$ are the critical points where the abs. max. of f would be attained. Evaluating $f(x,y)$ at these pts.

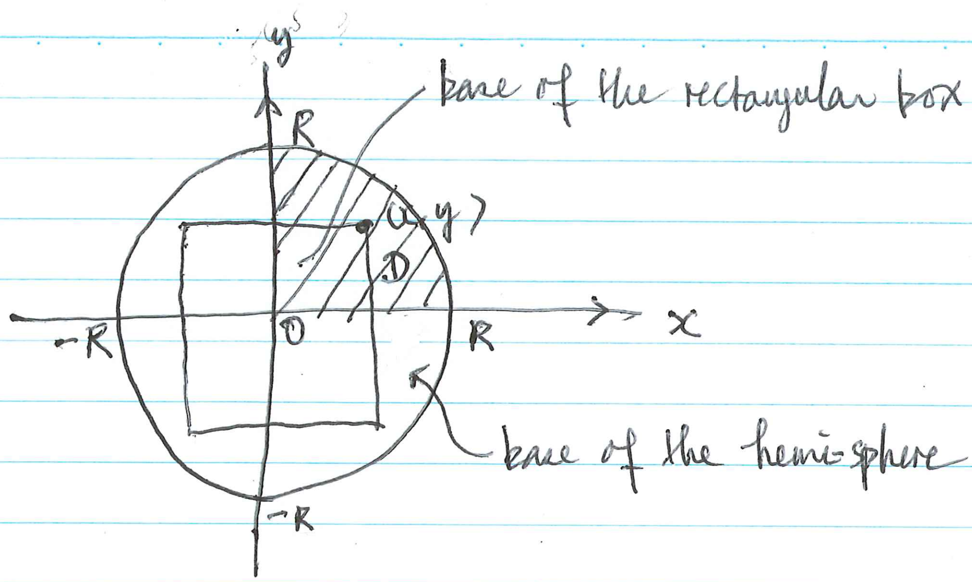
$$f(0,0) = 0, \; f(2,0) = 16/3, \; f(0,3) = 27, \; f(2,3) = \frac{97}{3}$$

Thus, $f(2,3) = \frac{97}{3}$ is our absolute maximum //.

Ex. What is the largest volume of a rectangular box that could be inscribed in a hemisphere of radius R? We assume one face of the box is on the base of the base of the hemisphere.

Solution

Without loss of generality, we may assume the center of the rectangular base of the box be at the origin of the x-y plane (see figure). Further, we let (x,y) be the corner of the base in the 1st quadrant.



We note that the equation of the hemi-spherical surface is $z=\sqrt{R^2-x^2-y^2}$, which is also the height of the rectangular box, the volume of the box is therefore,

$$f(x,y)=4xy\sqrt{R^2-x^2-y^2}\quad,\ (x,y)\in D$$

where D is the quarter circular region of radius R in the 1st quadrant. Since f is continuous over D, f must attain its absolute max and its abs. minimum over D. Further $f(x,y)=0$ along ∂D, the abs. max. must be attained at the interior of D where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist. Thus, at the point (x,y) where the absolute maximum of f is attained, we have

$$\frac{\partial f}{\partial x}=4y\sqrt{R^2-x^2-y^2}-\frac{4xy^2}{\sqrt{R^2-x^2-y^2}}=\frac{4y(R^2-y^2-2x^2)}{\sqrt{R^2-x^2-y^2}}=0$$

$\Rightarrow\ y^2+2x^2=R^2$ ———— (1) ($y=0$ could be rejected as mentioned before).

Similarly, we also have $\frac{\partial f}{\partial y}=0$ implying

$$x^2+2y^2=R^2 \text{ ———— (2)}$$

Solving (1) and (2) simultaneously, we have $x=y=\frac{R}{\sqrt{3}}$ which must be the point where the abs. max. of $f(x,y)$ is being attained (note that such result could actually be expected by symmetry argument).

The corresponding max. capacity of the box is therefore,

$$f\left(\frac{R}{\sqrt3},\frac{R}{\sqrt3}\right)=4\left(\frac{R}{\sqrt3}\right)^2\sqrt{R^2-\frac{2}{3}R^2}=\frac{4}{3}R^2\frac{R}{\sqrt3}=\frac{4R^3}{3\sqrt3}$$

//.

2nd derivative test for local max. and local min. when we have a smooth critical point

Thm. (2nd Derivative Test) Given $z=f(x,y)$, $(x,y)\in D$ and let $(a,b)\in D$ be a smooth critical point (f twice continuously differentiable) where $\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0$.

We define the discriminant $D_f(a,b)$ by

$$D_f(a,b) = f_{xx}(a,b) f_{yy}(a,b) - (f_{xy}(a,b))^2,$$

then we have

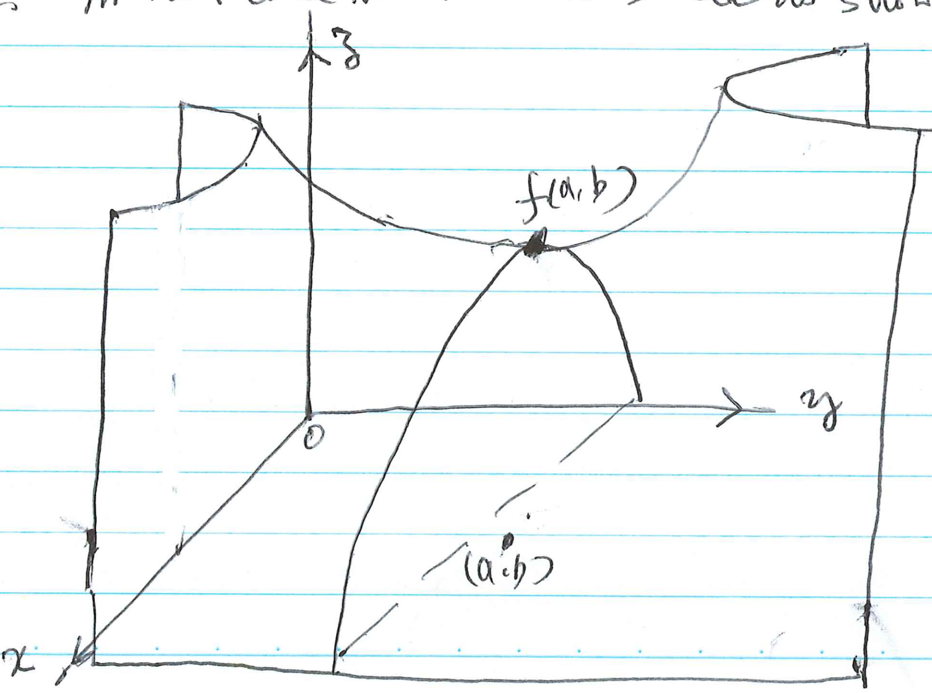
(i) If $D(a,b)>0$ and $f_{xx}(a,b)>0$, f is a local min. at (a,b)

(ii) If $D(a,b)>0$ and $f_{xx}(a,b)<0$, f is a local max at (a,b)

(iii) If $D(a,b)<0$, f has a saddle point at (a,b) (see figure below)

(iv) If $D(a,b)=0$, the test is inconclusive.

Remark: In case $D(a,b)>0$, $f_{xx}(a,b)$ and $f_{yy}(a,b)$ share the same sign. In case $D(a,b)<0$, $f_{xx}(a,b)$ and $f_{yy}(a,b)$ are opposite in sign. In that case we have a saddle as shown



$f_{xx}(a,b)>0$

$f_{yy}(a,b)<0$

Proof: Postpone until later.

Ex. Back to the earlier example $f(x,y) = \frac{8x^3}{3} + 4y^3 - x^4 - y^4$.

$\frac{\partial f(x,y)}{\partial x} = 8x^2 - 4x^3 = 0$, $\frac{\partial f(x,y)}{\partial y} = 12y^2 - 4y^3 = 0$

at $(0,0)$, $(2,0)$, $(0,3)$ and $(2,3)$

$$D_f = f_{xx} f_{yy} - (f_{xy})^2$$
$$= (16x - 12x^2)(24y - 12y^2) - (0)^2$$

$D_f(0,0) = D_f(2,0) = D_f(0,3) = 0$, test fails.

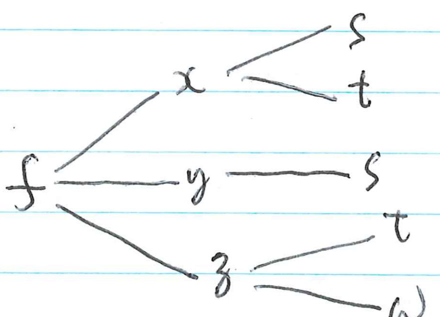
But at $(2,3)$, $f_{xx}(2,3) < 0$, $f_{yy} < 0$, $D_f(2,3) > 0$

$\therefore$ we have a local max at $(2,3)$ (which turns out to be an abs. max. as well).

Chain Rule for Partial Differentiation

Since we are dealing with functions of multi-variable, there is no single chain rule that could handle every possible case, but each individual case could be handled using a tree diagram.

Ex. Given $f(x,y,z)$ where x is a function of s & t, y a fct of s and z a function of t & w. As a result f is a function of s, t and w. The relationship could be expressed through a tree diagram.



we have $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{dy}{ds}$, $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t}$

$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial z} \frac{\partial z}{\partial w}$

Ex. $f(x,y,z) = xyz$
where $x = s^2 + t^2$, $y = st$, $z = tw^2$

Ex. Proof of the 2^{nd} derivative test

Let $f(x,y)$ have a critical pt. at (a,b) where f is twice continuous differentiable and $\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0$

Without loss of generality, we assume (a,b) is at $(0,0)$.

In order to show that f either attains a local max or a local min at (a,b), it boils down to showing that the curve of intersection between the surface $z=f(x,y)$ and any θ-plane (the vertical plane which makes an angle θ with the positive x-axis)

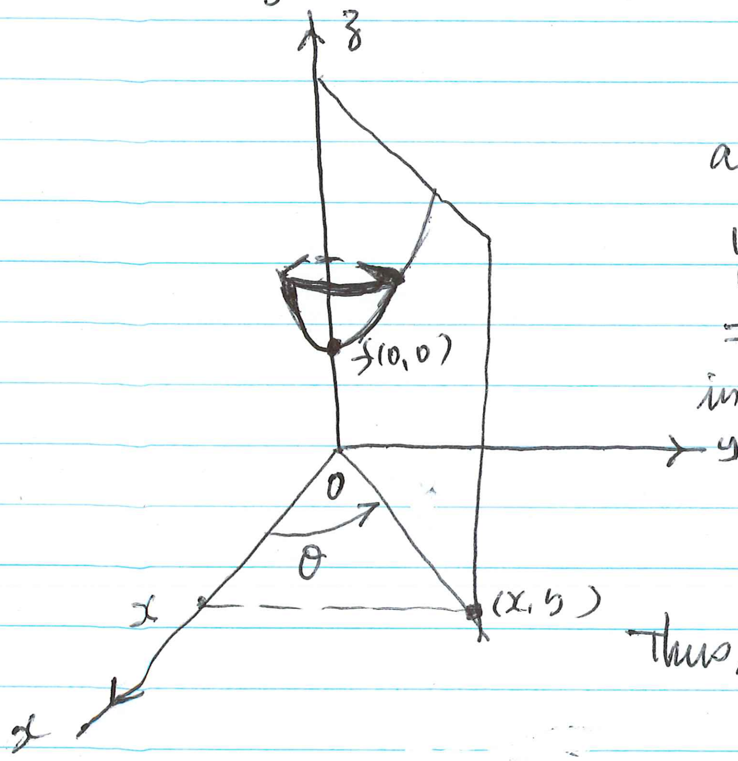


Diagram showing a local minimum at (0,0)

In this case, x and y are related through the eqn.

$y = \tan\theta\, x$

$\Rightarrow$ equation of curve of intersection is given by

$$z = f(x,y) = f(x, \tan\theta x)$$

Thus, f has the tree diagram

$$f \begin{cases} x \\ y - x \end{cases}$$

Indeed, set $g(x) = f(x, x\tan\theta)$, $\leftarrow$ equation of curve of intersection between the surface & the θplane

$$g'(x) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = f_x + f_y \tan\theta$$

Since $(0,0)$ is a critical point i.e. $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0)$, we must have

$$g'(0) = f_x(0,0) + f_y(0,0)\tan\theta = 0.$$

Consider now $g''(x) = f_{xx}(x, \tan\theta x) + 2\tan\theta f_{xy}(x, \tan\theta x)$
$+ f_{yy}(x, \tan\theta x)\tan^2\theta$

Here let us rule out the case $\theta = \pi/2$ & $3\pi/2$ (which actually corresponds to the case $x = 0$), so that $\tan\theta$ exists.

$$g''(0) = f_{xx}(0,0) + 2f_{xy}(0,0)\tan\theta + f_{yy}(0,0)\tan^2\theta$$

This is a quadratic function in $\tan\theta$ with discriminant

$$\Delta = 4f_{xy}(0,0)^2 - 4f_{xx}(0,0)f_{yy}(0,0)$$
$$= -4D_f(0,0).$$

Now that if $D_f(0,0) > 0$, $\Delta < 0 \Rightarrow g''(0) < 0$ or $g''(0) > 0$ for every $\theta \neq \pi/2$ and $3\pi/2$ and the sign of $g''(0)$ is determined by

$g''(0) > 0$ if $f_{yy}(0,0) > 0$ (alternatively $f_{xx}(0,0) > 0$)

$g''(0) < 0$ if $f_{yy}(0,0) > 0$ (alternatively $f_{xx}(0,0) < 0$)

Thus, $D_f(0,0) > 0$ and $f_{xx}(0,0) > 0 \Rightarrow$ a local minimum at $(0,0)$ in all θ-direction such that $\theta \neq \pi/2, 3\pi/2$

$D_f(0,0) > 0$ and $f_{xx}(0,0) < 0 \Rightarrow$ a local maximum at $(0,0)$ in all θ direction s.t. $\theta \neq \pi/2, 3\pi/2$

Finally, we consider the case when $\theta = \pi/2$ or $3\pi/2$, but this is the trivial case when $x = 0$ (i.e. the θ-plane co-incides with the y-z plane). We'll leave it as an exercise for the readers.

In the case when $D_f(0,0) < 0$, we have $g''(0) > 0$ along certain θ-directions & $g''(0) < 0$ along some other θ directions, this would correspond to the saddle point case //